HW6 MTH416/616 Spring 2025

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Ch 2 #2.48 Is solution of

$$x' = \begin{bmatrix} -2.2 & 0.3 & 1 & -0.4 \\ 1.5 & -3 & -1 & 0.3 \\ -1.2 & 0.8 & -3 & 0.7 \\ -0.1 & 0.3 & 0.3 & -1 \end{bmatrix} x$$

glob asy stable?

Soln: Yes, compute

$$\mu(A) = \max\{-2.2+|0.3|+|1|+|-0.4|,\ -3+|1.5|+|-1|+|0.3|,\ -3+|-1.2|+|0.8|+|0.7|,\ -1+|-0.1|+|0.3|+|0.3|\}$$

for one of the norms

$$= \max\{-0.5, -0.2, -0.3, -0.3\}$$

$$= -0.2 < 1$$ (thus by Cor 2.57, soln is glob asy stable)

2.52 Find Floquet multipliers for...

(i) $x' = (2\sin(3t))x$ — $\frac{2\pi}{3}$-periodic (ω)

$\sin(t)$ is 2π-periodic
$\sin(bt)$ is $\frac{2\pi}{b}$-periodic

Soln: Separation of variables

$$\Rightarrow \int \frac{1}{x}\,dx = 2\int \sin(3t)\,dt$$

$$\ln(x) = -\frac{2}{3}\cos(3t) + C$$

$$x = \tilde{c}\, e^{-\frac{2}{3}\cos(3t)}, \quad \tilde{c} = e^{C}$$

$\phi(t) \sim$ fund soln

$\Rightarrow$ Floquet multipliers are

$$\mu = \frac{\phi(\frac{2\pi}{3})}{\phi(0)} = \frac{e^{-\frac{2}{3}\overbrace{\cos(2\pi)}^{=1}}}{e^{-2/3}} = 1$$

(note: this is only Floquet mult $\sim$ notice the soln is $\frac{2\pi}{3}$-periodic!)
see Thm 2.70

(ii) $x' = \cos^2(t)x$ — trig identity: $\cos^2(t) = \frac{1+\cos(2t)}{2}$ — 2π-periodic

Soln: $\int \frac{1}{x}\,dx = \int \cos^2(t)\,dt$

$$\ln(x) = \int \frac{1+\cos(2t)}{2}\,dt = \frac{t}{2} + \frac{1}{4}\sin(2t) + C$$

$$\Downarrow$$

$$x(t) = \tilde{c}\, e^{\frac{t}{2}+\frac{1}{4}\sin(2t)}, \quad \tilde{c} = e^{C}$$

$\phi(t)$; π-periodic

Floquet multiplier:

$$\mu = \frac{\phi(\pi)}{\phi(0)} = \frac{e^{\frac{\pi}{2}+\frac{1}{4}\sin(2\pi)}}{e^{0}} = e^{\pi/2}$$

(iii) $x' = (-1+\sin(4t))x$ — $\frac{\pi}{2}$-periodic

Soln: $\int \frac{1}{x}\,dx = \int -1+\sin(4t)\,dt$

$$\ln(x) = -t - \frac{1}{4}\cos(4t) + C$$

$$x(t) = \tilde{c}\, e^{-t-\frac{1}{4}\cos(4t)}$$

$\phi(t)$

$$\Downarrow$$

$$\mu = \frac{\phi(\frac{\pi}{2})}{\phi(0)} = \frac{e^{-\frac{\pi}{2}-\frac{1}{4}\cos(2\pi)}}{e^{-0-1/4}} = \frac{e^{-\frac{\pi}{2}-\cancel{1/4}}}{\cancel{e^{-1/4}}} = e^{-\frac{\pi}{2}}$$

2.54 Find Floq. mult. for...

var of constants
$x' = px + q$
$$x(t) = e^{\int_{t_0}^{t} p(\tau)d\tau} x_0 + e^{\int_{t_0}^{t} p(\tau)d\tau}\int_{t_0}^{t} e^{-\int_{t_0}^{s} p(\tau)d\tau} q(s)\,ds$$

(i) $x' = \begin{bmatrix} -3+2\sin(t) & 0 \\ 0 & -1 \end{bmatrix} x$

Soln: $\begin{cases} x_1' = (-3+2\sin(t))x_1 \\ x_2' = -x_2 \end{cases}$ — 2π-periodic

var of constants on each ($q \equiv 0$)

$$e^{\int_0^t -3+2\sin(\tau)d\tau} = e^{-3t-2\cos(t)} \qquad e^{-\int_0^s p(\tau)d\tau} = e^{3s+2\cos(s)}$$

$$x_1(t) = x_0^{(1)} e^{-3t-2\cos(t)}$$

$$x_2(t) = x_0^{(2)} e^{-t}$$

$$\Downarrow$$

Soln is

$$x = \begin{bmatrix} x_0^{(1)} e^{-3t} e^{-2\cos(t)} \\ x_0^{(2)} e^{-t} \end{bmatrix} = \underbrace{\begin{bmatrix} e^{-3t}e^{-2\cos(t)} & 0 \\ 0 & e^{-t} \end{bmatrix}}_{\text{Fund soln } \phi(t)} \begin{bmatrix} x_0^{(1)} \\ x_0^{(2)} \end{bmatrix}$$

$$\phi^{-1}(t) = \begin{bmatrix} e^{3t+2\cos(t)} & 0 \\ 0 & e^{t} \end{bmatrix}$$

Floquet multi. are eigenvalues of

$$C = \phi^{-1}(0)\phi(2\pi)$$

$$= \begin{bmatrix} e^{0+2} & 0 \\ 0 & e^{0} \end{bmatrix}\begin{bmatrix} e^{-3(2\pi)-2\cos(2\pi)} & 0 \\ 0 & e^{-2\pi} \end{bmatrix}$$

$$= \begin{bmatrix} e^{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e^{-6\pi-2} & 0 \\ 0 & e^{-2\pi} \end{bmatrix} = \begin{bmatrix} e^{-6\pi} & 0 \\ 0 & e^{-2\pi} \end{bmatrix}$$

these are the e-vals!

(ii) $x' = \begin{bmatrix} -1+\cos(t) & 0 \\ \cos(t) & -1 \end{bmatrix} x$ — 2π-periodic

Soln: soln

$$x(t) = \begin{bmatrix} x_0^{(1)} e^{\sin(t)-t} \\ x_0^{(1)} e^{\sin(t)-t} + x_0^{(2)} e^{-t} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} e^{\sin(t)-t} & 0 \\ e^{\sin(t)-t} & e^{-t} \end{bmatrix}}_{\text{fund matrix } \phi}\begin{bmatrix} x_0^{(1)} \\ x_0^{(2)} \end{bmatrix} \qquad \phi \longrightarrow \phi^{-1}(t) = e^{t}\begin{bmatrix} e^{-\sin(t)} & 0 \\ -1 & 1 \end{bmatrix}$$

$\Rightarrow$ calculate

$$C = \phi^{-1}(0)\phi(2\pi)$$

$$= e^{0}\begin{bmatrix} e^{0} & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^{\sin(2\pi)-2\pi} & 0 \\ e^{\sin(2\pi)-2\pi} & e^{-2\pi} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^{-2\pi} & 0 \\ e^{-2\pi} & e^{-2\pi} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-2\pi} & 0 \\ -e^{-2\pi}+e^{-2\pi} & e^{-2\pi} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-2\pi} & 0 \\ 0 & e^{-2\pi} \end{bmatrix}$$

eigenvalues of C (i.e. Floquet multipliers)

(iii) $x' = \begin{bmatrix} -1 & 0 \\ \sin(t) & -1 \end{bmatrix} x$ — 2π-periodic

Soln

$$x(t) = \begin{bmatrix} x_1^{(0)} e^{-t} \\ -x_1^{(0)} e^{-t}\cos(t) + x_2^{(0)} e^{-t} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} e^{-t} & 0 \\ -e^{-t}\cos(t) & e^{-t} \end{bmatrix}}_{\phi}\begin{bmatrix} x_0^{(1)} \\ x_0^{(2)} \end{bmatrix} \qquad \phi \longrightarrow \phi^{-1}(t) = e^{t}\begin{bmatrix} 1 & 0 \\ \cos(t) & 1 \end{bmatrix}$$

$$C = \phi^{-1}(0)\phi(2\pi)$$

$$= e^{0}\begin{bmatrix} 1 & 0 \\ \cos(0) & 1 \end{bmatrix}\begin{bmatrix} e^{-2\pi} & 0 \\ -e^{-2\pi}\cos(2\pi) & e^{-2\pi} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{-2\pi} & 0 \\ -e^{-2\pi} & e^{-2\pi} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-2\pi} & 0 \\ e^{-2\pi}-e^{-2\pi} & e^{-2\pi} \end{bmatrix} = \begin{bmatrix} e^{-2\pi} & 0 \\ 0 & e^{-2\pi} \end{bmatrix}$$

eigenvalues of C = Floquet mults

(iv) $x' = \begin{bmatrix} -3+2\sin(t) & 0 \\ 0 & -1 \end{bmatrix} x$ — 2π-periodic

$$x(t) = \begin{bmatrix} x_1^{(0)} e^{-3t-2\cos(t)} \\ x_2^{(0)} e^{-t} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} e^{-3t-2\cos(t)} & 0 \\ 0 & e^{-t} \end{bmatrix}}_{\phi(t)}\begin{bmatrix} x_0^{(1)} \\ x_0^{(2)} \end{bmatrix} \qquad \phi(t) \longrightarrow \phi^{-1}(t) = \begin{bmatrix} e^{3t+2\cos(t)} & 0 \\ 0 & e^{t} \end{bmatrix}$$

$$C = \phi^{-1}(0)\phi(2\pi)$$

$$= \begin{bmatrix} e^{0+2} & 0 \\ 0 & e^{0} \end{bmatrix}\begin{bmatrix} e^{-6\pi-2} & 0 \\ 0 & e^{-2\pi} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-6\pi} & 0 \\ 0 & e^{-2\pi} \end{bmatrix}$$

Floq mults

2.55 Det stability of trivial soln for last four

(i) FM's are $e^{-6\pi}$ and $e^{-2\pi}$ $\sim$ both <1 $\Rightarrow$ asy stable

(ii) FM's are $e^{-2\pi}$ and $e^{-2\pi}$ $\sim$ " " $\to$ asy stable

(iii) FM's are ________ "

(iv) FM's are $e^{-6\pi}$ and $e^{-2\pi}$ ________ "

Grad students

Ch. 2 #2.53 Assume a is continuous on $\mathbb{R}$ and $a(t+\omega) = a(t)$ $\forall t \in \mathbb{R}$, $\omega > 0$.
Prove that $x' = a(t)x$ has all solns of form $x(t) = p(t)e^{rt}$
where $p(t) \neq 0$ is a continuously diff'bl fnct on $\mathbb{R}$ w/ period ω
and r is the average value of $a(t)$ on $[0, \omega]$, i.e. $r = \frac{1}{\omega}\int_0^{\omega} a(t)dt$

Show $\mu = e^{r\omega}$ is the Floquet multiplier
Show all solns periodic iff $\int_a^{\omega} a(t)dt = 0$.

Soln: $x' = a(t)x \xrightarrow{\text{var of const}} x(t) = x_0 e^{\int_0^t a(\tau)d\tau}$

assume form

$$\underset{\text{form we want}}{p(t)e^{rt}} = x(t) = \underset{\text{computed}}{x_0 e^{\int_0^t a(\tau)d\tau}}$$

$$\Downarrow$$

$$p(t) = x_0 e^{-rt} e^{\int_0^t a(\tau)d\tau}$$

Show p is ω-periodic:

$$p(t+\omega) = x_0 e^{-r(t+\omega)} e^{\int_0^{t+\omega} a(\tau)d\tau}$$

$$= x_0 e^{-rt} e^{-r\omega} e^{\int_0^t a(\tau)d\tau} e^{\int_t^{t+\omega} a(\tau)d\tau}$$

$$= p(t) e^{-r\omega} e^{\int_0^{\omega} a(\tau)d\tau}$$

a periodic $\Rightarrow \int_t^{t+\omega} a(\tau)d\tau$ is constant
Because it integrates over one period!
$\Downarrow$
so it equals $e^{\int_0^{\omega} a(\tau)d\tau}$

For p periodic, we require

$$e^{-r\omega} e^{\int_0^{\omega} a(\tau)d\tau} = 1$$

$$\Downarrow$$

$$-r\omega + \int^{\omega} a(\tau)d\tau = 0 \rightarrow r = \frac{1}{\omega}\int_0^{\omega} a(\tau)\,d\tau$$

Here, fund soln is

$$\phi(t) = p(t)e^{rt}$$

Floquet mult:

$$\mu = \frac{\phi(\omega)}{\phi(0)} = \frac{p(\omega)e^{r\omega}}{p(0)} = e^{r\omega}$$

$p(0) = p(\omega)$ since p periodic

Finally, all solns are ω-periodic
iff
$\mu = 1$
iff
$e^{r\omega} = 1$ iff $r\omega = 0$ iff $r = 0$ ($\omega > 0$)

#2.60 Show

$$x(t) = \begin{bmatrix} -e^{\frac{t}{2}}\cos(t) \\ e^{t/2}\sin(t) \end{bmatrix}$$

solves system — 2π-periodic

$$x' = \begin{bmatrix} -1+\frac{3}{2}\cos^2(t) & 1-\frac{3}{2}\cos(t)\sin(t) \\ -1-\frac{3}{2}\sin(t)\cos(t) & -1+\frac{3}{2}\sin^2(t) \end{bmatrix} x$$

and find Floq. mult.

Soln: Calculate

$$x' = \begin{bmatrix} -\frac{1}{2}e^{\frac{t}{2}}\cos(t) + e^{\frac{t}{2}}\sin(t) \\ \frac{1}{2}e^{t/2}\sin(t) + e^{t/2}\cos(t) \end{bmatrix} = e^{\frac{t}{2}}\begin{bmatrix} -\frac{1}{2}\cos(t)+\sin(t) \\ \frac{1}{2}\sin(t)+\cos(t) \end{bmatrix}$$

They match!

and calculate

$$\begin{bmatrix} -1+\frac{3}{2}\cos^2(t) & 1-\frac{3}{2}\cos(t)\sin(t) \\ -1-\frac{3}{2}\sin(t)\cos(t) & -1+\frac{3}{2}\sin^2(t) \end{bmatrix} x = \begin{bmatrix} -1+\frac{3}{2}\cos^2(t) & 1-\frac{3}{2}\cos(t)\sin(t) \\ -1-\frac{3}{2}\sin(t)\cos(t) & -1+\frac{3}{2}\sin^2(t) \end{bmatrix} e^{\frac{t}{2}}\begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix}$$

$$= e^{t/2}\begin{bmatrix} \cos(t) - \frac{3}{2}\cos^3(t) + \sin(t) - \frac{3}{2}\cos(t)\sin^2(t) \\ \cos(t) + \frac{3}{2}\sin(t)\cos^2(t) - \sin(t) + \frac{3}{2}\sin^3(t) \end{bmatrix}$$

$$= e^{t/2}\begin{bmatrix} \cos(t) - \frac{3}{2}\cos(t)\underbrace{(\cos^2(t)+\sin^2(t))}_{=1} + \sin(t) \\ \cos(t) + \frac{3}{2}\sin(t)(\cos^2(t)+\sin^2(t)) - \sin(t) \end{bmatrix}$$

$$= e^{t/2}\begin{bmatrix} -\frac{1}{2}\cos(t) + \sin(t) \\ \cos(t) + \frac{1}{2}\sin(t) \end{bmatrix}$$

Now, notice

$$x(t+2\pi) = \begin{bmatrix} -e^{\frac{(t+2\pi)}{2}}\cos(t+2\pi) \\ e^{\frac{t+2\pi}{2}}\sin(t+2\pi) \end{bmatrix} = e^{\pi}\begin{bmatrix} -e^{\frac{t}{2}}\cos(t) \\ e^{\frac{t}{2}}\sin(t) \end{bmatrix} = e^{\pi}x(t)$$

Thus $\mu_1 = e^{\pi}$ is a Floquet mult by Thm 2.70

We know by Thm 2.71 that

$$\mu_1\mu_2 = e^{\int_0^{2\pi} \operatorname{tr} A(t)\,dt}$$

$$\downarrow$$

$$e^{\pi}\mu_2 = e^{\int_0^{2\pi} \underbrace{(-1+\frac{3}{2}\cos^2(t)) + (-1+\frac{3}{2}\sin^2(t))}_{-2+\frac{3}{2} = -\frac{1}{2}}\,dt} = e^{-\int_0^{2\pi}\frac{1}{2}dt} = e^{-\pi}$$

$$\Downarrow$$

$$e^{\pi}\mu_2 = e^{-\pi}$$

$$\Downarrow$$

$$\mu_2 = e^{-2\pi}$$

eigenvalues of $A \longrightarrow \lambda = -\frac{1}{4} \pm \frac{\sqrt{7}}{4}i$

$\operatorname{Re}\lambda < 0$!