Ch 3 #2. Draw phase plane diagram for...

(2) $x' = -2xy^2 \leftarrow 0$, $y' = 2xy$

Solve: Divide eq options to get $\frac{dy}{dx}$

$\frac{y'}{x'} = \frac{2xy}{-2xy^2} \rightarrow \frac{dy}{dx} = -\frac{1}{y}$

$\int y\,dy = \int -dx$

$\frac{y^2}{2} = -x + C$

$x^2 + \frac{y^2}{2} = C$

$x = 0 \Rightarrow y = 0$ (one pt)

$y = 0 \Rightarrow x = 0$ (one pt)

$x = 0, y \neq 0$ (vert line)

$y = 0, x \neq 0$ (horiz line)

$(x,y) = (0,0)$ (one pt)

$x' = 0 \Rightarrow y = 0$ (horiz)

$y' = 0 \Rightarrow x = 0$ (vert)

$(x,y) = (0,0)$ (one pt)

$x' > 0 \Rightarrow y > 0$ (horiz right)

$y' > 0 \Rightarrow x < 0$ (vert up)

(x,y) in QII (i.e. $x<0$, $y>0$)

$x' < 0 \Rightarrow y < 0$ (horiz left)

$y' < 0 \Rightarrow x < 0$ (vert down)

(x,y) in QIII (i.e. $x<0$, $y<0$)

$x' > 0 \Rightarrow y < 0$ (horiz right)

$y' < 0 \Rightarrow x < 0$ (vert down)

(x,y) in QIV (i.e. $x>0$, $y<0$)

$x' < 0 \Rightarrow y > 0$ (horiz left)

$y' > 0 \Rightarrow x > 0$ (vert up)

(4) $x' = x^2 + y^2 + 1$, $y' = 1 + x^2$

Solve: Divide eq options to get $\frac{dy}{dx}$

$\frac{y'}{x'} = \frac{1+x^2}{x^2+y^2+1} \rightarrow \frac{dy}{dx} = \frac{1+x^2}{x^2+y^2+1}$

$\int (x^2+y^2+1)\,dy = \int (1+x^2)\,dx$

$x^2y + \frac{y^3}{3} + y = x + \frac{x^3}{3} + C$

$y = 0$ (horiz)

$x = 0$ (vert)

$x = y$ (one point)

$x = -y$ (one point)

$x > y$ (one right)

$x < y$ (one left)

$x > -y$ (one up)

$x < -y$ (one down)

Ch 3 #3. Find all eqpt pts of $\begin{cases} x' = y \\ y' = -\sin x \end{cases}$

Solve: Find eq pts algebraically

$\begin{cases} 0 = y \\ 0 = -\sin x \end{cases}$ (1)

$\sin x = 0$

$x = n\pi$, $n \in \mathbb{Z}$

$(n\pi, 0)$, $n \in \mathbb{Z}$

Then sketch phase plane diagram

$\frac{dy}{dx} = \frac{-\sin x}{y}$

$\int y\,dy = \int -\sin x\,dx$

$\frac{y^2}{2} = \cos x + C$

$y^2 = 2\cos x + C$

$y = \pm\sqrt{2\cos x + C}$

$C = 1 \Rightarrow y = \pm\sqrt{2\cos x + 1}$

$C = 2 \Rightarrow y = \pm\sqrt{2\cos x + 2}$

$C = 3 \Rightarrow y = \pm\sqrt{2\cos x + 3}$

$C = -1 \Rightarrow y = \pm\sqrt{2\cos x - 1}$

$C = -2 \Rightarrow y = \pm\sqrt{2\cos x - 2}$

$C = -3 \Rightarrow y = \pm\sqrt{2\cos x - 3}$

$\cos x \geq -1$

$2\cos x \geq -2$

$2\cos x + C \geq -2 + C$

$y^2 \geq -2 + C$

$C = 2 \Rightarrow y^2 \geq 0$ (all real y)

$C = 1 \Rightarrow y^2 \geq -1$ (all real y)

$C = 3 \Rightarrow y^2 \geq 1$ ($y \geq 1$ or $y \leq -1$)

$C = -1 \Rightarrow y^2 \geq -3$ (all real y)

$C = -2 \Rightarrow y^2 \geq -4$ (all real y)

$C = -3 \Rightarrow y^2 \geq -5$ (all real y)

$C = 2$ (all real y)

$C = 1$ (all real y)

$C = 3$ ($y \geq 1$ or $y \leq -1$)

$C = -1$ (all real y)

$C = -2$ (all real y)

$C = -3$ ($y \geq 1$ or $y \leq -1$)

P 3.5 Show $\begin{cases} x' = x + 2y \\ y' = -2x - y \end{cases}$ is a Hamiltonian system. Find H for which it has a level diagram.

Solve: System is Hamiltonian iff $\exists H$ s.t. $\begin{cases} x' = \frac{\partial H}{\partial y} \\ y' = -\frac{\partial H}{\partial x} \end{cases}$

So suppose we have $\frac{\partial H}{\partial y} = x + 2y$ and $-\frac{\partial H}{\partial x} = -2x - y$

$\frac{\partial H}{\partial x} = 2x + y$

$h(x,y) = \int x + 2y\,dy$

$= xy + 2\left(\frac{y^2}{2}\right) + C(x)$

$= xy + y^2 + C(x)$

assume C is only in terms of x with x and y

$\frac{\partial}{\partial x}$

$y + C'(x) = 2x + y$

$C'(x) = 2x$

$C(x) = x^2 + D$

Therefore, the Hamiltonian is $h(x,y) = xy + y^2 + x^2 + D$

Check: $h(x,y) = \text{constant}$

$xy + y^2 + x^2 = \text{constant}$

level curves slightly rotated ellipses

check if work?

$\frac{\partial h}{\partial y} = x + 2y$ ✓

$-\frac{\partial h}{\partial x} = -(y + 2x) = -2x - y$ ✓

P 3.7 Hamiltonian system or not? If yes, find h:

(1) $\begin{cases} x' = x^2 - 2x = \frac{\partial h}{\partial y} \\ y' = 2y - x = -\frac{\partial h}{\partial x} \end{cases}$ $\xrightarrow{\int dy}$ $h = x^2 - 2xy + C(x)$

$\frac{\partial}{\partial x}$

$-2y + C'(x) = x$

$C'(x) = x + 2y$

$C(x) = \frac{x^2}{2} + 2yx + D$

$\Rightarrow h(x,y) = x^2 - 2xy + \frac{x^2}{2} + 2yx + D$

$= \frac{3}{2}x^2 + D$

(2) $\begin{cases} x' = xy + y = \frac{\partial h}{\partial y} \\ y' = -(x+x^2) - \frac{x^2}{2} = -x - \frac{3}{2}x^2 = -\frac{\partial h}{\partial x} \end{cases}$ $\xrightarrow{\int dy}$ $h = \frac{xy^2}{2} + \frac{y^2}{2} + C(x)$

$\frac{\partial}{\partial x}$

$\frac{y^2}{2} + C'(x) = -x - \frac{3}{2}x^2$

$C'(x) = -\frac{x^2}{2} - x - \frac{x^3}{2}$ can't happen! Conclude that C depends on y, but it can't!

not Hamiltonian

(3) $\begin{cases} x' = x^2 - xy^2 = -\frac{\partial h}{\partial y} \\ y' = 2x^2y + 2xy = \frac{\partial h}{\partial x} \end{cases}$ $\xrightarrow{\int dy}$ $h = -2x^2y + \frac{xy^3}{3} + C(x)$

$\frac{\partial}{\partial x}$

$-4xy + \frac{y^3}{3} + C'(x) = 2x^2y + 2xy$

$C'(x) = 2x^2y + 6xy - \frac{y^3}{3}$

$\Rightarrow h(x,y) = -2x^2y + 3xy^2 - \frac{xy^3}{3} + D$

$= -\frac{1}{3}(6x^2y - 9xy^2 + xy^3) + D$

$= -\frac{1}{3}xy(6x - 9y + y^2) + D$

$= -\frac{1}{3}xy(y^2 - 9y + 6x) + D$

$= -\frac{1}{3}xy(y^2 - 3y - 6y + 6x) + D$

$= -\frac{1}{3}xy(y(y-3) - 6(y-x)) + D$

$= -\frac{1}{3}xy(y-3)(y-6x) + D$

$= -\frac{1}{3}xy(y-3)(y-6x) + D$

$= -\frac{1}{3}xy(y-3)(y-6x