HW8 MTH 416/616 Spring 2025

Friday, April 18, 2025 9:21 PM

#16 (i) $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} x+x^2-2xy \\ -y+xy \end{bmatrix} \overset{\text{set}}{=} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\rightarrow \begin{cases} x+x^2-2xy=0 & (i) \\ -y+xy=0 \rightarrow y(-1+x)=0 \end{cases}$

$y(-1+x)=0$ OR:

- $y=0 \xrightarrow{(i)} x+x^2-0=0$, $x(1+x)=0$ → $x=0$ → $(0,0)$; $x=-1$ → $(-1,0)$
- $x-1=0$, $x=1 \xrightarrow{(i)} 1+1^2-2y=0$, $2-2y=0$, $y=1$ → $(1,1)$

equilibrium points: $(0,0)$ $(-1,0)$ $(1,1)$

General Jacobian:

$$J = \begin{bmatrix} 1+2x-2y & -2x \\ y & -1+x \end{bmatrix}$$

Now

$J(0,0)=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ → evals $1,-1$ → unstable (because $\exists\lambda$ s.t. $Re(\lambda)>0$)

$J(-1,0)=\begin{bmatrix} -1 & 2 \\ 0 & -2 \end{bmatrix}$ → evals $\lambda=-2,-1$ → asymptotically stable (because $Re(\lambda)<0$ for each λ)

$J(1,1)=\begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}$ → evals $\lambda=\frac{1}{2}(1\pm i\sqrt{7})$ → unstable

(ii) $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -x+y+y^2-2xy \\ x \end{bmatrix} \overset{\text{set}}{=} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\rightarrow \begin{cases} -x+y+y^2-2xy=0 \\ x=0 \end{cases} \rightarrow y+y^2=0$

$y(1+y)=0$

$y=0$ or $y=-1$

→ eq pts: $(0,0)$, $(0,-1)$

$$J=\begin{bmatrix} -1-2y & 1+2y-2x \\ 1 & 0 \end{bmatrix}$$

$J(0,0)=\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ → $\lambda=\frac{1}{2}\underbrace{(\sqrt{5}-1)}_{>0}, -\frac{1}{2}(1+\sqrt{5})$ → unstable

$J(0,-1)=\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ → $\lambda=\frac{1}{2}(1\pm i\sqrt{3})$ → unstable

(iii) $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} x-y \\ x^2-1 \end{bmatrix} \overset{\text{set}}{=} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{cases} x-y=0 & (i) \\ x^2-1=0 \rightarrow x=\pm 1 \end{cases}$

- $+$, (i): $1-y=0$, $y=1$ → $(1,1)$
- $-$, (i): $-1-y=0$ → $y=-1$ → $(-1,-1)$

$$J=\begin{bmatrix} 1 & -1 \\ 2x & 0 \end{bmatrix}$$

$J(1,1)=\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$ → $\lambda=\frac{1\pm\sqrt{7}i}{2}$ → unstable

$J(-1,-1)=\begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix}$ → $\lambda=-1,2$ → unstable

#18 (i) $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} x^5-2x+y \\ -x \end{bmatrix} \overset{\text{set}}{=} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{cases} x^5-2x+y=0 & (i) \\ -x=0 \rightarrow x=0 \end{cases}$

$x=0 \xrightarrow{(i)} y=0$ → $(0,0)$

$$J=\begin{bmatrix} 5x^4-2 & 1 \\ -1 & 0 \end{bmatrix}$$

$J(0,0)=\begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}$ → evals $\lambda=-1,-1$ → asy stable

(ii) $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 4-4x^2-y^2 \\ 3xy \end{bmatrix} \overset{\text{set}}{=} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{cases} 4-4x^2-y^2=0 & (i) \\ 3xy=0 & (ii) \end{cases}$

(ii) → $3xy=0$ OR:

- $x=0 \xrightarrow{(i)} 4-y^2=0$, $y=\pm 2$ → $+$: $(0,2)$; $-$: $(0,-2)$
- $y=0 \xrightarrow{(i)} 4-4x^2=0$ → $x^2=1$ → $x=\pm 1$ → $+$: $(1,0)$; $-$: $(-1,0)$

$$J=\begin{bmatrix} -8x & -2y \\ 3y & 3x \end{bmatrix}$$

$J(0,2)=\begin{bmatrix} 0 & -4 \\ 6 & 0 \end{bmatrix}$ → evals: $\lambda=\pm 2i\sqrt{6}$ → not asy stable

$J(0,-2)=\begin{bmatrix} 0 & 4 \\ -6 & 0 \end{bmatrix}$ → $\lambda=\pm 2i\sqrt{6}$ → not asy stable

$J(1,0)=\begin{bmatrix} -8 & 0 \\ 0 & 3 \end{bmatrix}$ → $\lambda=-8,3$ → unstable

$J(-1,0)=\begin{bmatrix} 8 & 0 \\ 0 & -3 \end{bmatrix}$ → $\lambda=8,-3$ → unstable

#19 $\begin{cases} x'=-x-5y \\ y'=3x-y^3 \end{cases}$

$V=\alpha x^2+\beta y^2$

$\dot{V}=2\alpha xx'+2\beta yy'$

$=2\alpha x[-x-5y]+2\beta y[3x-y^3]$

$=-2\alpha x^2-10\alpha xy+6\beta yx-2\beta y^4$

Consider $-10\alpha xy+6\beta yx=0$

$xy\underbrace{[-10\alpha+6\beta]}_{\text{making this }=0}=0$

$\Downarrow$

$10\alpha=6\beta$

$\beta=\frac{10}{6}\alpha$

Take $\alpha=6 \rightarrow \beta=10$

With $\alpha=6, \beta=10$ we have

$\dot{V}=-12x^2-20y^4<0$ ✓

#20 $\begin{cases} x'=-x-y^3 \\ y'=x-y \end{cases}$

$V=\alpha x^2+\beta y^4$

$\dot{V}=2\alpha xx'+4\beta y^3y'$

$=2\alpha x(-x-y^3)+4\beta y^3(x-y)$

$=-2\alpha x^2\underbrace{-2\alpha xy^3+4\beta y^3x}_{\text{want to choose } \alpha,\beta \text{ so that}}-4\beta y^4$

$-2\alpha xy^3+4\beta y^3x=0$

$xy^3[-2\alpha+4\beta]=0$

$\alpha=2\beta$

Take $\beta=1$, $\alpha=2$

$\rightarrow \dot{V}=-4x^2-4y^4<0$

For grad students

#27 Show $(K,0)$ is asymp stable whenever $bK<c$ in system

$$\begin{cases} N'=rN(1-\frac{N}{K})-aNP \leftarrow \text{call this } f(N,P) \\ P'=bNP-cP \leftarrow g(N,P) \end{cases}$$

$r,K,a,b,c>0$

Soln:

$$J=\begin{bmatrix} \frac{\partial f}{\partial N} & \frac{\partial f}{\partial P} \\ \frac{\partial g}{\partial N} & \frac{\partial g}{\partial P} \end{bmatrix} = \begin{bmatrix} r(1-\frac{N}{K})+rN(-\frac{1}{K})-aP & -aN \\ bP & -c \end{bmatrix}$$

$J(\underset{N}{K},\underset{P}{0}) = \begin{bmatrix} r\cancel{(1-\frac{K}{K})}^{0}+rK(-\frac{1}{K})-0 & -aK \\ 0 & -c \end{bmatrix}$

$=\begin{bmatrix} -r & -aK \\ 0 & -c \end{bmatrix}$

$\downarrow$ evalues

$\lambda=-c,-r$

both negative $\Rightarrow$ asymptotically stable