HW9 MTH 416/616 Spring 2025

Saturday, April 19, 2025 11:04 AM

#34 Show that

$$(*)\quad \begin{cases} x' = -4x - y + x(x^2+y^2) \\ y' = x - 4y + y(x^2+y^2) \end{cases}$$

has a nontrivial periodic soln.

Soln: Use polar coords: $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases} \to x^2+y^2 = r^2$

Now

$$r^2 = x^2 + y^2$$

$\downarrow \frac{d}{dt}$

$$2rr' = 2xx' + 2yy'$$

$$\overset{(*)}{=} 2x\left(-4x - y + x(x^2+y^2)\right) + 2y\left(x - 4y + y(x^2+y^2)\right)$$

$$= -8x^2 - 2xy + 2x^2(r^2) + 2yx - 8y^2 + 2y^2(r^2)$$

$$= -8r^2 + 2r^4$$

Thus,

$$\boxed{r' = \frac{-8r^2+2r^4}{2r} = -4r + r^3}$$

Now for θ, start with

$$\tan(\theta) = \frac{y}{x}$$

$\downarrow \frac{d}{dt}$

$$\theta' \sec^2(\theta) = \frac{xy' - yx'}{x^2}$$

$$\overset{(*)}{=} \frac{x\left(x - 4y + y(x^2+y^2)\right) - y\left(-4x - y + x(x^2+y^2)\right)}{x^2}$$

$$= \frac{x^2 - 4xy + xyr^2 + 4xy + y^2 - xyr^2}{x^2}$$

$$= \frac{r^2}{x^2} \overset{\text{polar def}}{=} \frac{r^2}{r^2\cos^2(\theta)} = \sec^2(\theta)$$

$\downarrow$ div by $\sec^2(\theta)$

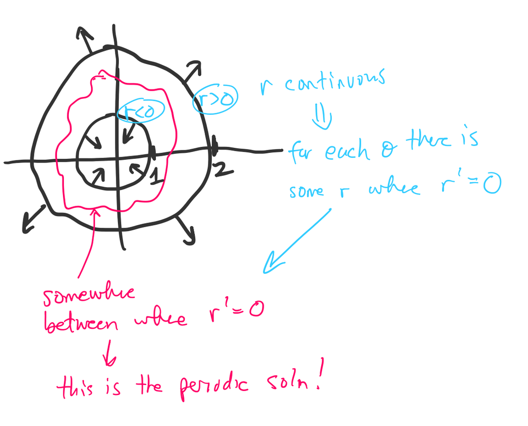
$$\boxed{\theta' = 1}$$

So our system is

$$\begin{cases} r' = -4r + r^3 \\ \theta' = 1 \end{cases}$$

Notice at $r = 1$, $r' < 0$

at $r = 2$, $r' > 0$



#36 Show

$$\begin{cases} x' = -x - y + x(3x^2+y^2) \\ y' = x - y + y(3x^2+y^2) \end{cases} \leftarrow \text{does it actually have one?}$$

has a limit cycle.

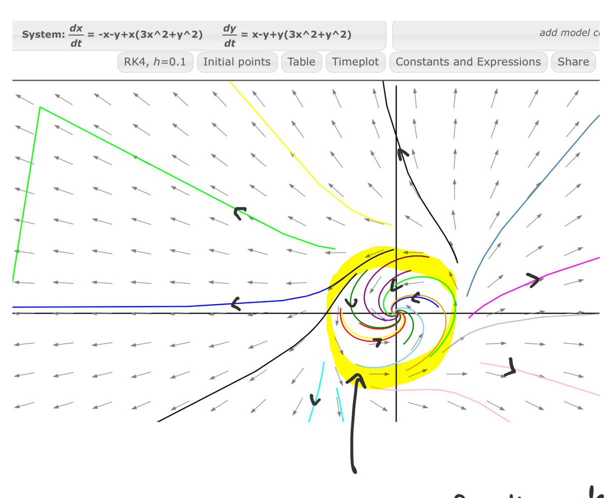
Soln: First find equilibria:

$$\begin{cases} 0 = x' = -x - y + x(3x^2+y^2) & (i) \\ 0 = y' = x - y + y(3x^2+y^2) & (ii) \end{cases}$$

$\downarrow$ computer

only real soln is $(x,y) = (0,0)$

all others are complex!



Candidate for lim cycle but solns don't converg to it!

Solns inside the orbit go to equilibrium & solns outside shoot off!

No limit cycle here...
But soln on left shows how to find if one exists

Now put into polar:

$$r^2 = x^2 + y^2$$

$$2rr' = 2xx' + 2yy'$$

$$= 2x\left[-x - y + x(3x^2+y^2)\right] + 2y\left[x - y + y(3x^2+y^2)\right]$$

$$= -2x^2 - 2xy + 2x^2(3x^2+y^2) + 2xy - 2y^2 + 2y^2(3x^2+y^2)$$

$$= -2(x^2+y^2) + 2(3x^2+y^2)(x^2+y^2)$$

$$= 2(x^2+y^2)\left[-1 + \underbrace{3x^2+y^2}_{x^2+y^2+2x^2}\right]$$

$$= 2r^2\left[-1 + r^2 + 2x^2\right]$$

$$= 2r^2\left[-1 + r^2 + 2r^2\cos^2(\theta)\right]$$

$$= 2r^2\left[-1 + r^2\left[1 + 2\cos^2(\theta)\right]\right]$$

Also

$$\tan(\theta) = \frac{y}{x}$$

$\downarrow \frac{d}{dt}$

$$\sec^2(\theta)\theta' = \frac{xy' - yx'}{x^2} = \frac{x\left(x - y + y(3x^2+y^2)\right) - y\left(-x - y + x(3x^2+y^2)\right)}{x^2}$$

$$= \frac{x^2 - xy + xy(3x^2+y^2) + xy + y^2 - xy(3x^2+y^2)}{x^2}$$

$$= \frac{r^2}{x^2} = \frac{1}{\cos^2(\theta)} = \sec^2(\theta)$$

$\downarrow$

$$\theta' = 1$$

Thus our system is

$$\begin{cases} r' = -r + r^3 + 2r^3\cos^2(\theta) \\ \theta' = 1 \end{cases}$$

$-1 \le \cos(\theta) \le 1$

$\downarrow$

$0 \le \cos^2(\theta) \le 1$

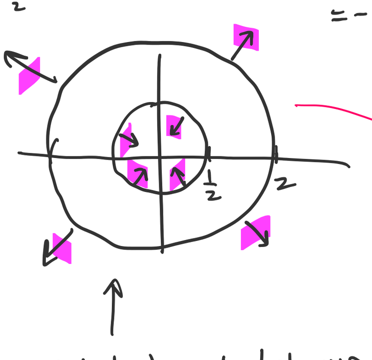
At $r = 2$

$$r'\big|_{r=2} = -2 + 8 + 16\cos^2(\theta) \ge 6$$

but at $r = \frac{1}{2}$

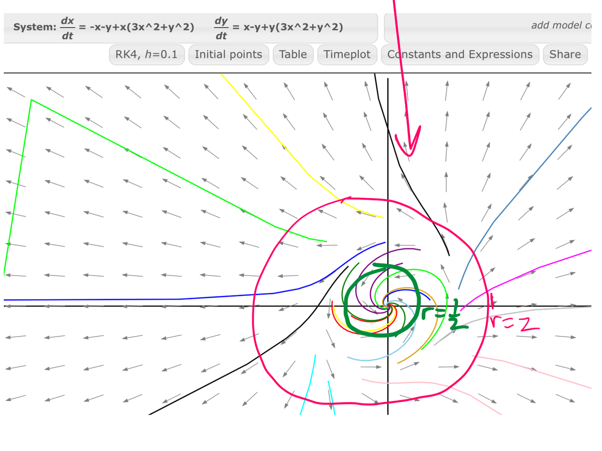
$$r'\big|_{r=\frac{1}{2}} = -\frac{1}{2} + \frac{1}{8} + \frac{2}{8}\cos^2(\theta) \le -\frac{1}{2} + \frac{1}{8} + \frac{1}{4}$$

$$= -\frac{4}{8} + \frac{1}{8} + \frac{2}{8} = -\frac{1}{8} < 0$$



Not limit cycle behavior
the arrows going wrong way

But this picture consistent with the numerical soln



#3.40 Show

$$\begin{cases} x' = x - y - x\left(x^2 + \frac{3}{2}y^2\right) \\ y' = x + y - y\left(x^2 + \frac{1}{2}y^2\right) \end{cases} \xrightarrow{\text{computer}} \text{equilibria: only } (0,0)$$

has a limit cycle

Soln: Use polar form:

$$r^2 = x^2 + y^2$$

$\downarrow \frac{d}{dt}$ and div by 2

$$rr' = xx' + yy'$$

$$= x\left[x - y - x\left(x^2 + \tfrac{3}{2}y^2\right)\right] + y\left[x + y - y\left(x^2 + \tfrac{1}{2}y^2\right)\right]$$

$$= x^2 - xy - x^2\left(x^2 + \tfrac{3}{2}y^2\right) + yx + y^2 - y^2\left(x^2 + \tfrac{1}{2}y^2\right)$$

$$= r^2 - x^2\left(x^2 + y^2 + \tfrac{1}{2}y^2\right) - y^2\left(x^2 + y^2 - \tfrac{1}{2}y^2\right)$$

$$= r^2 - x^2r^2 - \tfrac{1}{2}x^2y^2 - y^2r^2 + \tfrac{1}{2}y^4$$

$$= r^2 - r^4 - \tfrac{1}{2}y^2\left[x^2 - y^2\right]$$

$$= r^2 - r^4 - \tfrac{1}{2}r^2\sin^2(\theta)\left[r^2\cos^2(\theta) - r^2\sin^2(\theta)\right]$$

$$= r^2 - r^4 - \tfrac{1}{2}r^4\sin^2(\theta)\underbrace{\left[\cos^2(\theta) - \sin^2(\theta)\right]}_{=\cos(2\theta) \text{ by trig}}$$

$$= r^2 - r^4 - \tfrac{1}{2}r^4\sin^2(\theta)\cos(2\theta)$$

Thus

$$r' = r - r^3 - \tfrac{1}{2}r^3\sin^2(\theta)\cos(2\theta)$$

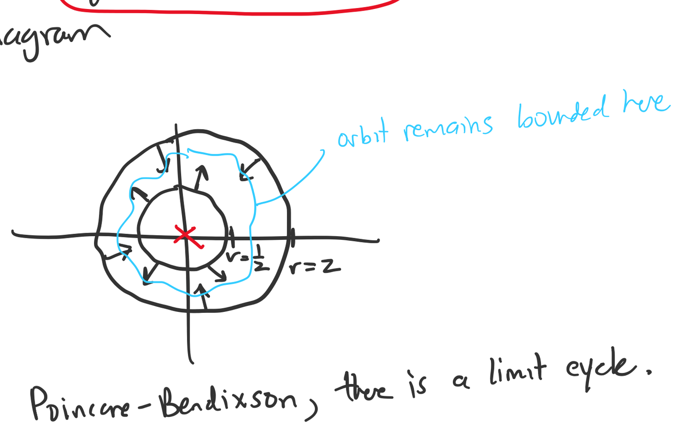
$\sin^2(\theta) \ge 0$

at $r = \frac{1}{2}$: $r'\big|_{r=\frac{1}{2}} = \frac{1}{2} - \frac{1}{8} - \frac{1}{16}\sin^2(\theta)\cos(2\theta) \ge \frac{1}{2} - \frac{1}{8} > 0$ (points away from $(0,0)$)

while at $r = 2$: $r'\big|_{r=2} = 2 - 8 - 8\sin^2(\theta)\cos(2\theta) \le 2 - 8 - 8 < 0$ (points in toward $(0,0)$)

$\sin^2(\theta) \le 1$, $\cos(2\theta) \le 1$

So we have only equilibrium at $(0,0)$ and diagram



Therefore by Poincare-Bendixson, there is a limit cycle.

#5.1 Put into self-adjoint form:

(i) (Legendre's eqt)

$$(1-t^2)x'' - 2tx' + n(n+1)x = 0, \quad t \in (-1,1)$$

$\downarrow$ div by $1-t^2$

$$x'' - \frac{2t}{1-t^2}x' + \frac{n(n+1)}{1-t^2}x = 0$$

Let $\mu = e^{-\int \frac{2t}{1-t^2}dt} = e^{\int \frac{1}{u}du} = e^{\ln(u)} = e^{\ln(1-t^2)} = 1-t^2$

$u = 1-t^2$

$du = -2t\,dt$

Now mult by μ: (seems pointless, we got back original, but won't always be this way)

$$\underbrace{(1-t^2)x'' - 2tx'}_{} + n(n+1)x = 0$$

$$\boxed{\left((1-t^2)x'\right)' + n(n+1)x = 0}$$

(ii) (Chebyshev's eqt)

$$(1-t^2)x'' - tx' + n^2x = 0$$

$\downarrow$

$$x'' - \frac{t}{1-t^2}x' + \frac{n^2}{1-t^2}x = 0$$

$$\mu = e^{-\int \frac{t}{1-t^2}dt} = e^{\frac{1}{2}\int \frac{1}{u}du} = e^{\frac{1}{2}\ln(u)} = e^{\ln(\sqrt{u})} = \sqrt{u} = \sqrt{1-t^2}$$

$u = 1-t^2$

$du = -2t\,dt$

$\frac{1}{2}du = -t\,dt$

$\downarrow$ mult by μ

$$\underbrace{\sqrt{1-t^2}\,x'' - \frac{t}{\sqrt{1-t^2}}x'}_{} + \frac{n^2}{\sqrt{1-t^2}}x = 0$$

$$\boxed{\left(\sqrt{1-t^2}\,x'\right)' + \frac{n^2}{\sqrt{1-t^2}}x = 0}$$

(iii) (Laguerre's eqt) $tx'' + (1-t)x' + ax = 0$

$\downarrow$

$$x'' + \frac{1-t}{t}x' + \frac{a}{t}x = 0$$

$$\mu = e^{\int \frac{1-t}{t}} = e^{\int \frac{1}{t} - 1\,dt} = e^{\ln(t) - t} = te^{-t}$$

mult by μ

$$te^{-t}x'' + \left(\tfrac{1}{t} - 1\right)te^{-t}x' + \tfrac{a}{t}te^{-t}x = 0$$

$$\underbrace{te^{-t}x'' + (1-t)e^{-t}x'}_{} + ae^{-t}x = 0$$

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+14

$$\boxed{\left(te^{-t}x'\right)' + ae^{-t}x = 0}$$

(iv) (Hermite's eqt) $x'' - 2tx' + 2nx = 0$

$$\mu = e^{\int -2t} = e^{-t^2}$$

$$\underbrace{e^{-t^2}x'' - 2te^{-t^2}x'}_{} + 2ne^{-t}x = 0$$

$$\boxed{\left(e^{-t^2}x'\right)' + 2ne^{-t}x = 0}$$

(v) $x'' + \frac{2}{3+t}x' + \frac{\lambda}{(3+t)^2}x = 0$

$$\mu = e^{\int \frac{2}{3+t}} = e^{2\ln(3+t)} = e^{\ln((3+t)^2)} = (3+t)^2$$

$$\underbrace{(3+t)^2x'' + 2(3+t)x'}_{} + \lambda x = 0$$

$$\left((3+t)^2x'\right)' + \lambda x = 0$$