Quiz 6 MTH 335 Fall 2026

Monday, October 12, 2026 4:20 PM

Solve

$$\sin^2(t)y'' - 2\sin(t)\cos(t)y' + (1+\cos^2(t))y = 0$$

given that $y_1(t) = \sin(t)$ is a soln.

Soln: Since y_1 is a soln, it means

$$\sin^2(t)y_1'' - 2\sin(t)\cos(t)y_1' + (1+\cos^2(t))y = 0.$$

We consider a second soln y_2 of form $y_2(t) = u(t)y_1(t)$ for some unknown function u.

Compute

$$y_2' = u'y_1 + uy_1'$$

$$y_2'' = u''y_1 + u'y_1' + u'y_1' + uy_1''$$

$$= u''y_1 + 2u'y_1' + uy_1''$$

(by doing this, we are declaring that y_2 is a solution! our job is to find u that justifies that claim)

Now substitute these y_2, y_2', and y_2'' into the ODE:

$$\sin^2(t)\left[u''y_1 + 2u'y_1' + uy_1''\right] - 2\sin(t)\cos(t)\left[u'y_1 + uy_1'\right] + (1+\cos^2(t))uy_1 = 0$$

$$u''\left(\sin^2(t)y_1\right) + u'\left(2y_1'\sin^2(t) - 2\sin(t)\cos(t)y_1\right) + u\underbrace{\left(y_1''\sin^2(t) - 2\sin(t)\cos(t)y_1' + (1+\cos^2(t))y_1\right)}_{=0 \text{ since } y_1 \text{ is a solution!}} = 0$$

Since $y_1(t) = \sin(t)$ we have $y_1'(t) = \cos(t)$ and $y_1''(t) = -\sin(t)$.

Thus we have

$$u''\left(\sin^3(t)\right) + u'\left(2\cos(t)\sin^2(t) - 2\sin^2(t)\cos(t)\right) = 0$$

($2\cos(t)\sin^2(t) - 2\sin^2(t)\cos(t) = 0$)

$$(u'')(\sin^3(t)) = 0$$

$$\downarrow$$

$$u'' = 0$$

$$\downarrow \int$$

$$u' = A \quad \text{(for some constant } A\text{)}$$

$$\downarrow \int$$

$$u = At + B \quad \text{(for some } B\text{)}$$

So our 2nd soln is

$$y_2 = uy_1 = (At+B)\sin(t)$$

$$= At\sin(t) + B\sin(t)$$

- At: ignore constant (we will recover it later)
- $A\sin(t)$: take as y_2 (i.e. take $B=0, A=1$)
- $B\sin(t)$: already covered by our y_1

So now we have independent solns

$$y_1(t) = \sin(t) \qquad y_2(t) = t\sin(t)$$

$\Rightarrow$ general soln is

$$\boxed{y(t) = c_1\sin(t) + c_2 t\sin(t)}$$